CONVERGENCE PROPERTIES OF FOURIER SERIES*

BY

OTTO SZÁSZ

I. Introduction

1. In our previous papers† we were concerned with Fourier series

$$f(\theta) \sim \frac{a_0}{2} + \sum_{\nu=1}^{\infty} (a_\nu \cos \nu\theta + b_\nu \sin \nu\theta)$$

the coefficients of which satisfy the conditions

$$\nu a_\nu \geqq -K, \ \nu b_\nu \geqq -K, \ \nu \geqq 1, \ K \text{ a non-negative constant.}$$

These conditions characterize a special case of "slowly oscillating" series, and it is natural to generalize our results in this direction. In what follows the knowledge of our previous papers is not presupposed, except for the proof of Lemma 5 below.

2. We start with some preliminary notions. A series of real terms $\sum_0^\infty c_\nu$, or the corresponding sequence of its partial sums $\{s_\nu\}$, is called slowly oscillating from below if to any positive ϵ there corresponds an integer $N=N(\epsilon)$ and a positive number $\delta=\delta(\epsilon)$ such that

$$s_{n+k} - s_n > -\epsilon, \ n > N, \ 0 < k < \delta(n+1).$$

It is readily seen that this condition is equivalent to

$$\lim_{\delta\to 0} \liminf_{n\to\infty} \min_{0<k<\delta(n+1)} (s_{n+k} - s_n) \geqq 0.$$

We denote by (A) the class of such series, or sequences.

A series is called slowly oscillating from below in the generalized sense if there exist two positive numbers q and δ such that

$$s_{n+k} - s_n > -q; \quad n = 0, 1, 2, \cdots; \quad 0 < k < \delta(n+1),$$

which is equivalent to

$$\liminf_{n\to\infty} \min_{0<k<\delta(n+1)} (s_{n+k} - s_n) > -\infty.$$

The class of such series will be denoted by $(\overline{A})$. It is obvious that $(A) \subset (\overline{A})$.

* Presented to the Society, September 6, 1934; received by the editors August 4, 1934.

† See [5, 6]. The numbers in brackets refer to the list of the author's previous papers at the end of the present paper.

These notions are important for converse theorems of the theory of summability.*

Let (A') be the class of series satisfying the condition $\liminf_{n\to\infty} nc_n > -\infty$. Then we have $(A') \subset (A) \subset (\overline{A})$. If $\{\delta_\nu\}$ is any bounded sequence of non-negative numbers and $\sum_0^\infty c_\nu \subset (A')$ then also $\sum_0^\infty c_\nu\delta_\nu \subset (A')$. This property was used in our papers [5, 6]. An analogous but larger class of series is obtained if we observe that, in order that the series $\sum_0^\infty c_\nu\delta_\nu \subset (A)$ for an arbitrary bounded and non-negative sequence $\{\delta_\nu\}$, it is necessary and sufficient that it belong to (A) in the special case

$$\delta_n = 1, \text{ or } 0, \text{ according as } c_n < 0, \text{ or } \geqq 0,$$

in other words that the series $\sum_0^\infty \frac{1}{2}(c_\nu - |c_\nu|) \subset (A)$. The class of such series we denote by (B). Thus (B) is the class of series such that $\sum_0^\infty c_\nu\delta_\nu \subset (A)$ whenever $\sum_0^\infty c_\nu \subset (A)$. We have the inclusion relation $(A') \subset (B) \subset (A) \subset (\overline{A})$.

Similarly we say that $\sum_0^\infty c_\nu \subset (\overline{B})$ if $\sum_0^\infty \frac{1}{2}(c_\nu - |c_\nu|) \subset (\overline{A})$. The class $(\overline{B})$ is identical to the class of series $\sum_0^\infty c_\nu$ such that $\sum_0^\infty c_\nu\delta_\nu \subset (\overline{A})$ for an arbitrary bounded and non-negative sequence $\{\delta_\nu\}$, whenever $\sum_0^\infty c_\nu \subset (\overline{A})$. It is plain that $(A') \subset (B) \subset (\overline{B}) \subset (\overline{A})$.

3. In the present paper we propose to prove the following theorems.

THEOREM 1. *Let* $\sum_1^\infty a_\nu r^\nu \cos \nu\theta$, $0 \leqq r < 1$, *represent a harmonic function and let*

(i) $$\sum_{\nu=1}^\infty a_\nu \subset (\overline{B}),$$

(ii) $$\sum_{\nu=1}^\infty a_\nu r^\nu = O(1), \quad 0 \leqq r < 1.$$

Then also

(ii′) $$\sum_{\nu=1}^n a_\nu = O(1) \qquad (n = 1, 2, 3, \cdots).$$

Conversely, the assumption (ii′) *implies* (ii). *If, in addition to* (i) *and* (ii) *we assume that for a fixed* $\theta = \theta_0$

(iii) $$\sum_{\nu=1}^\infty a_\nu r^\nu \cos \nu\theta_0 = O(1), \quad 0 \leqq r < 1,$$

then also

(iii′) $$\sum_{\nu=1}^n a_\nu \cos \nu\theta_0 = O(1) \qquad (n = 1, 2, 3, \cdots).$$

* Cf. [2], p. 332; [3], p. 30; [4], p. 326.

Conversely, (iii′) *implies* (iii). *Finally, if*

$$\phi(\theta) \sim \sum_{\nu=1}^{\infty} a_\nu \cos \nu\theta$$

is a Fourier series and

$$\sum_{\nu=1}^{n} a_\nu \cos \nu\theta_0 = O(1) \qquad (n = 1, 2, 3, \cdots),$$

then

$$\int_0^h \{\phi(\theta_0 + t) + \phi(\theta_0 - t)\} dt = O(h) \text{ as } h \to 0.$$

THEOREM 2. *Let* $\sum_1^\infty a_\nu \subset (B)$ *and* $\sum_1^\infty a_\nu r^\nu \to s$ *as* $r \to 1-0$. *Then* $\sum_1^\infty a_\nu$ *converges to* s.* *If in addition, for a fixed* $\theta = \theta_0$,

$$\sum_{\nu=1}^{\infty} a_\nu r^\nu \cos \nu\theta_0 \to s(\theta_0) \text{ as } r \to 1 - 0,$$

then $\sum_1^\infty a_\nu \cos \nu\theta_0$ *converges to* $s(\theta_0)$. *Finally, if*

$$\phi(\theta) \sim \sum_{\nu=1}^{\infty} a_\nu \cos \nu\theta$$

is a Fourier series and $\sum_1^\infty a_\nu \cos \nu\theta_0$ *converges to* $s(\theta_0)$, *then*

$$(2h)^{-1} \int_0^h \{\phi(\theta_0 + t) + \phi(\theta_0 - t)\} dt \to s(\theta_0) \text{ as } h \to 0.$$

THEOREM 3. *Let*

$$\omega(\theta) \sim \sum_{\nu=1}^{\infty} b_\nu \sin \nu\theta$$

be a Fourier series and $\sum_1^\infty b_\nu \subset (\overline{B})$. *Let*

$$\int_0^h \omega(t) dt = O(h) \text{ as } h \to 0.$$

Then

$$\sum_{\nu=1}^{n} \nu b_\nu = O(n) \qquad (n = 1, 2, 3, \cdots).$$

If, in addition,

$$\sum_{\nu=1}^{\infty} b_\nu r^\nu \sin \nu\theta_0 = O(1), \quad 0 \leqq r < 1,$$

* This is a well known result of R. Schmidt.

then

$$\sum_{\nu=1}^{n} b_\nu \sin \nu\theta_0 = O(1) \qquad (n = 1, 2, 3, \cdots).$$

Conversely, if

$$\sum_{\nu=1}^{n} b_\nu \sin \nu\theta_0 = O(1) \qquad (n = 1, 2, 3, \cdots),$$

then

$$\int_0^h \{\omega(\theta_0 + t) + \omega(\theta_0 - t)\} dt = O(h) \text{ as } h \to 0.$$

THEOREM 4. *Let* $\sum_1^\infty b_\nu \subset (B)$ *and*

$$\frac{2}{h}\int_0^h \omega(t) dt \to d \text{ as } h \to 0.*$$

Then

$$\frac{1}{n}\sum_{\nu=1}^{n} \nu b_\nu \to \frac{d}{\pi} \text{ as } n \to \infty.$$

If, in addition,

$$\sum_{\nu=1}^{\infty} b_\nu r^\nu \sin \nu\theta_0 \to s(\theta_0) \text{ as } r \to 1 - 0,$$

then $\sum_1^\infty b_\nu \sin \nu\theta_0$ *converges to* $s(\theta_0)$. *Conversely, if* $\sum_1^\infty b_\nu \sin \nu\theta_0$ *converges, then*

$$\frac{2}{h}\int_0^h \{\omega(\theta_0 + t) + \omega(\theta_0 - t)\} dt \to s(\theta_0) \text{ as } h \to 0.$$

In proving these theorems we are using methods analogous to those of our papers [5, 6]. Similar theorems hold with respect to the uniform boundedness or uniform convergence in a given interval $\alpha \leqq \theta \leqq \beta$. In the hypotheses of our theorems the sums $\sum_1^\infty \frac{1}{2}(a_\nu - |a_\nu|)$, $\sum_1^\infty \frac{1}{2}(b_\nu - |b_\nu|)$ may be replaced by the sums $\sum_1^\infty \frac{1}{2}(-a_\nu - |a_\nu|)$, $\sum_1^\infty \frac{1}{2}(-b_\nu - |b_\nu|)$ respectively.

Analogous theorems may also be stated for double Fourier series, as well as for almost periodic functions or for Fourier integrals.

The treatment of the cosine series is conspicuously simpler than that of the sine series, the reason being that to the value $\theta = 0$ in the first case there

* The quantity d may be interpreted as the generalized jump of $\omega(\theta)$ at $\theta = 0$.

corresponds the series $\sum_1^\infty a_\nu$ while, in the second case, all terms of the series vanish.*

II. The cosine series

1. **Proof of Theorem 1.** We begin by establishing some lemmas.

Lemma 1. *Put*

$$\sum_{\nu=1}^{n} c_\nu = s_n, \quad \sum_{\nu=1}^{n} \nu c_\nu = v_n, \quad v_0 = s_0 = c_0 = 0,$$

and assume that there exist two numbers, $p>0$ and $\mu>0$, such that

$$s_{n+k} - s_n \geqq -p, \quad 1 \leqq k < 1 + \mu(n+1) \qquad (n = 0, 1, 2, \cdots). \tag{1}$$

Then

$$v_{n+k} - v_n \geqq -p(n+k) \tag{2}$$

and

$$v_n > -p\frac{1+\mu}{\mu}n, \quad n > 0.† \tag{3}$$

Since

$$v_n = \sum_{\nu=0}^{n} (s_n - s_\nu), \quad v_{n+k} = \sum_{\nu=0}^{n+k} (s_{n+k} - s_\nu)$$

we have

$$v_{n+k} - v_n = (n+1)(s_{n+k} - s_n) + \sum_{\nu=n+1}^{n+k} (s_{n+k} - s_\nu) \geqq -p(n+1) - p(k-1),$$

which is the desired inequality (2). Now put

$$n_\nu = [n(1+\mu)^{-\nu}] \qquad (\nu = 0, 1, 2, \cdots),$$

so that

$$v_n = \sum_{\nu=0}^{\infty} (v_{n_\nu} - v_{n_{\nu+1}}),$$

where only a finite number of terms are different from zero. In view of the obvious inequality $n_\nu - n_{\nu+1} < 1 + \mu(1 + n_{\nu+1})$ we have

$$v_n \geqq -p\sum_{\nu=0}^{\infty} n_\nu \geqq -pn\sum_{\nu=0}^{\infty} (1+\mu)^{-\nu} = -p\frac{1+\mu}{\mu}n,$$

* The results of this paper were communicated in the author's seminar during the spring term at the Massachusetts Institute of Technology, and also at the colloquium at Brown University May 18, 1934.

† Cf. [2], p. 333; [4], pp. 326–327.

which proves (3).

LEMMA 2. *Under the assumptions* (1) *and*

$$\left|\sum_{\nu=1}^{\infty} c_\nu x^\nu\right| \leqq M, \quad 0 \leqq x < 1, \tag{4}$$

we have

$$|s_n| < M(1+8e)\frac{2+\mu}{\mu} + p\left(1 + 4e\frac{(2+\mu)(1+\mu)}{\mu^2}\right).$$

We set

$$\sum_{\nu=1}^{\infty} c_\nu x^\nu = P(x);$$

then from

$$\frac{v_n}{n+1} = s_n - (n+1)^{-1}\sum_{\nu=0}^{n} s_\nu$$

it follows that

$$\sum_{\nu=1}^{\infty} \frac{v_\nu}{\nu+1} x^\nu = (1-x)^{-1}P(x) - x^{-1}\int_0^x P(t)(1-t)^{-2}dt,$$

and, by (4),

$$\left|\sum_{\nu=1}^{\infty} \frac{v_\nu}{\nu+1} x^\nu\right| \leqq 2M(1-x)^{-1}.$$

Hence

$$\sum_{\nu=1}^{\infty}\left(\frac{v_\nu}{\nu+1} + p\frac{1+\mu}{\mu}\right)x^\nu \leqq 2M(1-x)^{-1} + p\frac{1+\mu}{\mu}x(1-x)^{-1}.$$

The coefficients of the power series of the left-hand member are positive, by (3). Consequently, for $x = 1 - 1/(n+1)$,

$$\begin{aligned}\left(1-\frac{1}{n+1}\right)^n \sum_{\nu=1}^{n}\left(\frac{v_\nu}{\nu+1} + p\frac{1+\mu}{\mu}\right) & \\ < \sum_{\nu=1}^{n}\left(\frac{v_\nu}{\nu+1} + p\frac{1+\mu}{\mu}\right)&\left(1-\frac{1}{n+1}\right)^\nu \\ < \left(2M + p\frac{1+\mu}{\mu}\right)&(n+1),\end{aligned}$$

whence

$$\sum_{\nu=1}^{n} \frac{v_\nu}{\nu+1} < \left(2M + p\frac{1+\mu}{\mu}\right)\left(1+\frac{1}{n}\right)^n (n+1)$$
$$< \left(2M + p\frac{1+\mu}{\mu}\right) e(n+1).$$

On the other hand, again by (3),

$$\sum_{\nu=1}^{n} \frac{v_\nu}{\nu+1} > -pn\frac{1+\mu}{\mu}, \qquad n > 0,$$

which yields the result

$$\left|\sum_{\nu=1}^{n} \frac{v_\nu}{\nu+1}\right| < e\left(2M + p\frac{1+\mu}{\mu}\right)(n+1), \qquad n > 0.$$

Now, on setting

$$(n+1)^{-1}\sum_{\nu=0}^{n} s_\nu = \sigma_n, \qquad \sum_{\nu=1}^{n} \frac{v_\nu}{\nu+1} = u_n,$$

we have

$$x^{-1}\int_0^x P(t)(1-t)^{-2}dt = \sum_{\nu=1}^{\infty} \sigma_\nu x^\nu,$$

$$(1-x)x^{-1}\int_0^x P(t)(1-t)^{-2}dt = \sum_{\nu=1}^{\infty} (\sigma_\nu - \sigma_{\nu-1})x^\nu = \sum_{\nu=1}^{\infty} \frac{v_\nu}{\nu(\nu+1)} x^\nu,$$

and*

$$(1-x)x^{-1}\int_0^x P(t)(1-t)^{-2}dt - \sigma_n = (1-x)\sum_{\nu=1}^{\infty} \frac{u_\nu}{\nu+1} x^\nu - \frac{u_n}{n+1}$$
$$+ \sum_{\nu=1}^{n} \frac{u_\nu}{\nu(\nu+1)}(x^\nu - 1) + \sum_{\nu=n+1}^{\infty} \frac{u_\nu}{\nu(\nu+1)} x^\nu.$$

By (4)

$$\left|(1-x)x^{-1}\int_0^x P(t)(1-t)^{-2}dt\right| \leqq M.$$

On combining this with the preceding inequality we have

$$|\sigma_n| < M + 2e\left(2M + p\frac{1+\mu}{\mu}\right)$$
$$+ e\left(2M + p\frac{1+\mu}{\mu}\right)\left\{n(1-x) + \frac{1}{n+1}\frac{x^{n+1}}{1-x}\right\},$$

* Cf. [1], Theorem 1, with νa_ν replaced by $v_\nu/(\nu+1)$.

whence, for $x=1-1/(n+1)$,

$$|\sigma_n| < M + 4e\left(2M + p\frac{1+\mu}{\mu}\right) \equiv M_1, \qquad n \geqq 1.$$

Consequently

$$s_n = \frac{v_n}{n+1} + \sigma_n > -p\frac{1+\mu}{\mu} - M_1, \qquad n \geqq 1.$$

On the other hand we have*

$$s_n = \sigma_{n+k} + \frac{n+1}{k}(\sigma_{n+k} - \sigma_n) - \frac{1}{k}\sum_{\nu=1}^{k}(s_{n+\nu} - s_n),$$

whence, for $\mu(n+1) \leqq k < 1 + \mu(n+1)$,

$$s_n < M_1 + 2M_1/\mu + p.$$

Lemma 2 follows at once by combining these two inequalities for s_n.

We now pass on to the proof of Theorem 1. Under the hypotheses (i) and (ii) Lemma 2 implies

$$s_n = \sum_{\nu=1}^{n} a_\nu = O(1) \qquad (n = 1, 2, 3, \cdots).$$

Conversely, if this condition is satisfied then

$$\sum_{\nu=1}^{\infty} a_\nu x^\nu = (1-x)\sum_{\nu=1}^{\infty} s_\nu x^\nu = O(1), \qquad 0 \leqq x < 1.$$

Furthermore, the series $\sum_1^\infty a_\nu(1-\cos\nu\theta_0)$ also $\subset(\overline{A})$. Then, if (iii) is satisfied, by Lemma 2,

$$\sum_{\nu=1}^{n} a_\nu(1-\cos\nu\theta_0) = O(1), \text{ and } \sum_{\nu=1}^{n} a_\nu \cos\nu\theta_0 = O(1) \qquad (n = 1, 2, 3, \cdots).$$

The converse is proved by the same argument as before.

To prove the last statement of the theorem we observe that

$$(2h)^{-1}\int_0^h \{\phi(\theta_0 + t) + \phi(\theta_0 - t)\}dt = \sum_{\nu=1}^{\infty} a_\nu \cos\nu\theta_0(\sin\nu h)/(\nu h).$$

By assumption $\sum_1^\infty(a_\nu - |a_\nu|) \subset (\overline{A})$, and we have proved that $\sum_1^n a_\nu = O(1)$. Hence the series $-\sum_1^\infty a_\nu$ as well as $-\sum_1^\infty |a_\nu| \subset (\overline{A})$. This means that, for suitably chosen p and μ,

$$\sum_{\nu=n+1}^{n+k} |a_\nu| \leqq p, \qquad 1 \leqq k < 1 + \mu(n+1), \qquad n \geqq 0.$$

* Cf. [3], p. 31.

Then, by Lemma 1,

$$\sum_{\nu=1}^{n} \nu \mid a_\nu \mid < np \frac{1+\mu}{\mu}, \qquad n > 0.$$

Now put

$$\sum_{\nu=1}^{\infty} a_\nu \cos \nu\theta_0 (\sin \nu h)/(\nu h) = \sum_{\nu=1}^{n} a_\nu \cos \nu\theta_0 + \sum_{\nu=1}^{n} a_\nu \cos \nu\theta_0((\sin \nu h)/(\nu h) - 1)$$

$$+ \sum_{\nu=n+1}^{\infty} a_\nu \cos \nu\theta_0 (\sin \nu h)/(\nu h) \equiv S_1 + S_2 + S_3.$$

In the previous argument we have proved the existence of a constant G such that

$$| S_1 | \leqq G.$$

From

$$0 \leqq 1 - (\sin \nu h)/(\nu h) \leqq \tfrac{1}{6}\nu^2 h^2$$

we have

$$(5) \qquad | S_2 | \leqq \tfrac{1}{6} h^2 \sum_{\nu=1}^{n} \nu^2 | a_\nu | \leqq \tfrac{1}{6} n h^2 \sum_{\nu=1}^{n} \nu | a_\nu | < p \frac{1+\mu}{6\mu} n^2 h^2.$$

Finally, on writing $\tau_{nk} = \sum_{n+1}^{n+k} | a_\nu | /\nu$ we have

$$\tau_{nk} \leqq (n+1)^{-1} \sum_{\nu=n+1}^{n+k} | a_\nu | \leqq p/(n+1), \qquad 1 \leqq k < 1 + \mu(n+1).$$

Let $n_0, n_1, \cdots$ be a sequence of integers such that

$$(1+\mu)n_{\nu-1} + \mu \leqq n_\nu < (1+\mu)(1+n_{\nu-1}), \qquad n_0 = n.$$

Then

$$\sum_{\nu=n+1}^{n_k} | a_\nu | /\nu = \sum_{\nu=n_0+1}^{n_1} + \cdots + \sum_{\nu=n_{k-1}+1}^{n_k} \leqq p \sum_{\nu=0}^{k-1} (n_\nu + 1)^{-1}$$

$$\leqq \frac{p}{n+1} \sum_{\nu=0}^{k-1} (1+\mu)^{-\nu} < \frac{p(1+\mu)}{(n+1)\mu},$$

since

$$n_\nu + 1 \geqq (1+\mu)(n_{\nu-1}+1) \geqq (1+\mu)^\nu (n+1).$$

Thus

$$\sum_{\nu=n+1}^{\infty} | a_\nu | /\nu \leqq \frac{p(1+\mu)}{(n+1)\mu}$$

and

$$|S_3| \leqq h^{-1} \sum_{\nu=n+1}^{\infty} |a_\nu|/\nu \leqq \frac{p(1+\mu)}{h(n+1)\mu}. \tag{6}$$

On choosing $n=[1/h]$, $h<1$, we finally obtain

$$\left| (2h)^{-1} \int_0^h \{\phi(\theta_0+t)+\phi(\theta_0-t)\} dt \right| < G + \frac{7}{6} p \frac{1+\mu}{\mu}.$$

2. **Proof of Theorem 2.** By hypothesis $\sum_1^\infty (a_\nu - |a_\nu|) \subset (A)$. Since $\sum_1^\infty a_\nu$ converges, $-\sum_1^\infty |a_\nu|$ also $\subset (A)$. Hence $\sum_1^\infty a_\nu \cos \nu\theta \subset (A)$ for every θ. If now

$$\sum_{\nu=1}^{\infty} a_\nu r^\nu \cos \nu\theta_0 \to s(\theta_0) \quad \text{as} \quad r \to 1-0,$$

the theorem of R. Schmidt mentioned above implies

$$\sum_{\nu=1}^{\infty} a_\nu \cos \nu\theta_0 = s(\theta_0).$$

It remains to prove the last statement of Theorem 2. We write

$$\begin{aligned} (2h^{-1}) &\int_0^h \{\phi(\theta_0+t)+\phi(\theta_0-t)\} dt - s(\theta_0) \\ &= \left(\sum_{\nu=1}^{n} a_\nu \cos \nu\theta_0 - s(\theta_0) \right) + \sum_{\nu=1}^{n} a_\nu \cos \nu\theta_0 \left(\frac{\sin \nu h}{\nu h} - 1 \right) \\ &\qquad + \sum_{\nu=n+1}^{\lambda_n} a_\nu \cos \nu\theta_0 \frac{\sin \nu h}{\nu h} + \sum_{\nu=\lambda_n+1}^{\infty} a_\nu \cos \nu\theta_0 \frac{\sin \nu h}{\nu h} \\ &= Z_0 + Z_1 + Z_2 + Z_3, \end{aligned}$$

where $\lambda_n > n$ will be fixed later. For a given ϵ, $0<\epsilon<1$, choose $N=N(\epsilon)$ so that

$$|Z_0| < \epsilon^2, \qquad n > N. \tag{7}$$

From (5) and (6) we have

$$|Z_1| < p \frac{1+\mu}{\mu} n^2 h^2, \qquad |Z_3| \leqq \frac{p(1+\mu)}{h(\lambda_n+1)\mu}.$$

Let $h_0 = h_0(\epsilon)$ be so small that

$$h_0 < \epsilon, \qquad [\epsilon/h_0] > N,$$

and put

$$n = [\epsilon/h], \qquad \lambda_n = [\pi/(\epsilon h)], \qquad h < h_0,$$

so that

$$nh \leqq \epsilon, \qquad 1 + \lambda_n > \pi/(\epsilon h).$$

Then

$$|Z_1| < p\,\frac{1+\mu}{\mu}\,\epsilon^2, \qquad |Z_3| < \frac{p(1+\mu)}{\pi\mu}\,\epsilon.$$

To estimate Z_2 subdivide the range $0<\nu h \leqq \lambda_n h < \pi/\epsilon$ into subintervals in each of which $(\sin \nu h)/(\nu h)$ is monotone (as function of ν), the number of these subintervals being not greater than $1+[1/\epsilon]$. An easy application of the partial summation formula together with (7) will show that

$$|Z_2| < (2/\epsilon)2\epsilon^2 = 4\epsilon.$$

To complete the proof of Theorem 2 it remains to allow $\epsilon \to 0$.

III. The sine series

1. Proof of Theorem 3. We shall need some additional lemmas.

LEMMA 3. *Let*

$$\omega(\theta) \sim \sum_{\nu=1}^{\infty} b_\nu \sin \nu\theta.$$

If

$$\int_0^h \omega(t)dt = O(h) \;\; as \;\; h \to 0, \tag{8}$$

then

$$\sum_{\nu=1}^{\infty} \nu b_\nu r^\nu = O((1-r)^{-1}) \;\; as \;\; r \to 1-0.$$

We have*

$$\sum_{\nu=1}^{\infty} \nu b_\nu r^\nu = -\pi^{-1}\int_0^\pi \omega(t)\,\frac{d}{dt}\,p(r,t)dt, \qquad 0 \leqq r < 1,$$

where

$$p(r,t) = (1-r^2)/\Delta, \quad \Delta = 1 - 2r\cos t + r^2 = (1-r)^2 + 4r\sin^2\frac{t}{2},$$
$$\frac{d}{dt}\,p(r,t) = -2r(1-r^2)\sin t/\Delta^2.$$

On setting

$$\psi(t) = \int_0^t \omega(\tau)d\tau$$

* Cf. [5], formula (21).

and integrating by parts we have

$$\sum_{\nu=1}^{\infty} \nu b_\nu r^\nu = -\pi^{-1} 2r(1-r^2) \int_0^\pi \psi(t) \frac{d}{dt} (\sin t\, \Delta^{-2}) dt.$$

In view of (8) an easy calculation yields

$$\sum_{\nu=1}^{\infty} \nu b_\nu r^\nu = O\left\{(1-r)^2 \int_0^\infty t\{(1-r)^2 + t^2\}^{-2} dt\right\} = O((1-r)^{-1}).$$

LEMMA 4. *If* (8) *holds and* $\sum_1^\infty b_\nu \subset (\overline{A})$, *then*

$$v_n = \sum_{\nu=1}^{n} \nu b_\nu = O(n) \qquad (n = 1, 2, 3, \cdots).$$

By Lemma 3

$$\sum_{\nu=1}^{\infty} \nu b_\nu r^\nu = O((1-r)^{-1}),$$

while, by Lemma 1,

$$v_\nu + p \frac{1+\mu}{\mu} \nu > 0 \qquad (\nu = 1, 2, 3, \cdots), \tag{3'}$$

with suitably chosen p and μ. Hence

$$\sum_{\nu=1}^{\infty} \left(v_\nu + p \frac{1+\mu}{\mu} \nu\right) r^\nu = O((1-r)^{-2}),$$

$$\sum_{\nu=1}^{n} \left(v_\nu + p \frac{1+\mu}{\mu} \nu\right) = O\left\{ \sum_{\nu=1}^{n} \left(v_\nu + p \frac{1+\mu}{\mu} \nu\right) \left(1 - \frac{1}{n}\right)^\nu \right\} = O(n^2).$$

Thus

$$V_n = \sum_{\nu=1}^{n} v_\nu = O(n^2) \qquad (n = 1, 2, 3, \cdots).$$

On the other hand the relation

$$kv_n = \frac{kV_{n+k}}{n+k+1} + (n+1)\left(\frac{V_{n+k}}{n+k+1} - \frac{V_n}{n+1}\right) - \sum_{\nu=1}^{k} (v_{n+\nu} - v_n)$$

gives, in view of (2),

$$v_n < Cn,$$

where C is a generic notation for a constant, not necessarily the same in all formulas where it occurs. On combining this with (3′) we obtain a proof of Lemma 4, and also of the first statement of Theorem 3.

Now, if $\sum_1^\infty b_\nu \subset (\overline{B})$, then $\sum_1^\infty b_\nu(1-\sin \nu\theta) \subset (\overline{A})$. If, in addition,

$$\sum_{\nu=1}^{\infty} b_\nu r^\nu \sin \nu\theta_0 = O(1), \qquad 0 \leqq r < 1,$$

then, as in the proof of Lemma 2,

$$\sum_{\nu=1}^{\infty} (\nu+1)^{-1}(b_1 \sin \theta_0 + \cdots + \nu b_\nu \sin \nu\theta_0)r^\nu = O((1-r)^{-1}),$$

while, by Lemma 4,

$$\sum_{\nu=1}^{\infty} \nu b_\nu r^\nu = O((1-r)^{-1}), \qquad \sum_{\nu=1}^{\infty} (b_1 + \cdots + \nu b_\nu)r^\nu = O((1-r)^{-2}),$$

and

$$\sum_{\nu=1}^{\infty} (\nu+1)^{-1}(b_1 + \cdots + \nu b_\nu)r^\nu = O((1-r)^{-1}).$$

Thus

$$\sum_{\nu=1}^{\infty} (\nu+1)^{-1}\{b_1(1-\sin\theta_0) + \cdots + \nu b_\nu(1-\sin \nu\theta_0)\}r^\nu = O((1-r)^{-1}).$$

We can now apply Lemma 1 obtaining

$$S_n = \sum_{\nu=1}^{n} \nu b_\nu(1-\sin \nu\theta_0) > -Cn, \qquad n \geqq 1. \tag{9}$$

Using the same argument as in the proof of Lemma 4 we get

$$R_n = \sum_{\nu=1}^{n} S_\nu/(\nu+1) = O(n) \qquad (n = 1, 2, 3, \cdots).$$

Writing for simplicity $R_0=0$ we have

$$\sum_{\nu=1}^{n} S_\nu = \sum_{\nu=1}^{n} (\nu+1)S_\nu/(\nu+1) = \sum_{\nu=1}^{n} (\nu+1)(R_\nu - R_{\nu-1})$$

$$= (n+1)R_n - \sum_{\nu=1}^{n-1} R_\nu = O(n^2),$$

whence, again as in Lemma 4,

$$S_n < Cn.$$

On combining these results we have

$$\sum_{\nu=1}^{n} \nu b_\nu \sin \nu\theta_0 = O(n) \qquad (n = 1, 2, 3, \cdots).$$

This, together with

$$\sum_{\nu=1}^{\infty} b_\nu r^\nu \sin \nu\theta_0 = O(1), \qquad 0 \leqq r < 1,$$

gives

$$\sum_{\nu=1}^{n} b_\nu \sin \nu\theta_0 = O(1) \qquad (n = 1, 2, 3, \cdots),$$

by an argument analogous to that used in the proof of Lemma 2.

Conversely, assume

$$\sum_{\nu=1}^{n} b_\nu \sin \nu\theta_0 = O(1) \qquad (n = 1, 2, 3, \cdots),$$

and write

$$\begin{aligned} (2h)^{-1}\int_0^h \{\omega(\theta_0+t)+\omega(\theta_0-t)\}\,dt &= \sum_{\nu=1}^{\infty} b_\nu \sin \nu\theta_0 \frac{\sin \nu h}{\nu h} \\ &= \sum_{\nu=1}^{n} b_\nu \sin \nu\theta_0 + \sum_{\nu=1}^{n} b_\nu \sin \nu\theta_0 \left(\frac{\sin \nu h}{\nu h} - 1\right) + \sum_{\nu=n+1}^{\infty} b_\nu \sin \nu\theta_0 \frac{\sin \nu h}{\nu h} \\ &\equiv J_1 + J_2 + J_3. \end{aligned}$$

Then, by assumption,

$$J_1 = O(1).$$

For J_2 we obtain the estimate

$$|J_2| \leqq \tfrac{1}{6}h^2 \sum_{\nu=1}^{n} \nu^2 |b_\nu| \leqq \tfrac{1}{6}nh^2 \sum_{\nu=1}^{n} \nu |b_\nu|.$$

But, by Lemma 4, $\sum_1^n \nu b_\nu = O(n)$ and, since $\sum_1^\infty b_\nu \subset (\overline{B})$, by Lemma 1, $\sum_1^n \nu(b_\nu - |b_\nu|) > -Cn$. Thus

$$g_n = \sum_{\nu=1}^{n} \nu |b_\nu| = O(n)$$

and

$$J_2 = h^2 O(n^2).$$

Finally, on putting $t_{nk} = \sum_{n+1}^{n+k} |b_\nu|/\nu$ we have

$$
\begin{aligned}
t_{nk} &= \sum_{\nu=n+1}^{n+k} (g_\nu - g_{\nu-1})\nu^{-2} \\
&= -g_n(n+1)^{-2} + g_{n+k}(n+k)^{-2} + \sum_{\nu=n+1}^{n+k-1} g_\nu(\nu^{-2} - (\nu+1)^{-2}) \\
&= O\left(\frac{1}{n}\right).
\end{aligned}
$$

Hence $\sum_{n+1}^{\infty} |b_\nu|/\nu = O(1/n)$ and

$$
|J_3| \leqq h^{-1} \sum_{\nu=n+1}^{\infty} |b_\nu|/\nu = h^{-1}O\left(\frac{1}{n}\right).
$$

On choosing $n = [1/h]$, $h<1$, we get immediately

$$
\int_0^h \{\omega(\theta_0 + t) + \omega(\theta_0 - t)\} dt = O(h).
$$

2. **Proof of Theorem 4.** We first state two additional lemmas.

Lemma 5. *Let*

$$
\omega(\theta) \sim \sum_{\nu=1}^{\infty} b_\nu \sin \nu\theta.
$$

If the limit

$$
\lim_{h\to 0} \frac{2}{h} \int_0^h \omega(t)dt = d \tag{10}
$$

exists, then

$$
(1-r)\sum_{\nu=1}^{\infty} \nu b_\nu r^\nu \to \frac{d}{\pi} \quad as \quad r \to 1 - 0.^*
$$

Lemma 6. *If* (10) *holds and* $\sum_1^\infty b_\nu \subset (A)$, *then*

$$
\frac{v_n}{n} = n^{-1} \sum_{\nu=1}^{n} \nu b_\nu \to \frac{d}{\pi} \quad as \quad n \to \infty.
$$

By Lemma 5,

$$
(1-r)\sum_{\nu=1}^{\infty} \nu b_\nu r^\nu \to \frac{d}{\pi} \text{ as } r \to 1,
$$

while, by Lemma 1,

$$
v_\nu + p\frac{1+\mu}{\mu}\nu] > 0 \qquad (\nu = 1, 2, 3, \cdots),
$$

* This was proved in our paper [5], Theorem 5.

with suitably chosen p and μ. Hence

$$(1-r)^2\sum_{\nu=1}^{\infty} v_\nu r^\nu \to \frac{d}{\pi},$$

$$(1-r)^2\sum_{\nu=1}^{\infty}\left(v_\nu + p\frac{1+\mu}{\mu}\nu\right)r^\nu \to \frac{d}{\pi} + p\frac{1+\mu}{\mu}, \text{ as } r\to 1.$$

By a theorem of Hardy and Littlewood,

$$\sum_{\nu=1}^{n}\left(v_\nu + p\frac{1+\mu}{\mu}\nu\right) \sim \left(\frac{d}{\pi} + p\frac{1+\mu}{\mu}\right)\frac{n^2}{2},$$

so that

$$V_n = \sum_{\nu=1}^{n} v_\nu \sim dn^2/(2\pi) \text{ as } n\to\infty.$$

On the other hand the relation

$$v_n/n = (m-n)^{-1}(V_m/m - V_n/n) + V_m/(mn) - n^{-1}(m-n)^{-1}\sum_{\nu=n+1}^{m}(v_\nu - v_n),$$
$$m > n > 0,$$

combined with

$$V_n = n^2(d/(2\pi) + \epsilon_n), \qquad \epsilon_n \to 0,$$

gives

$$\begin{aligned} v_n/n &= (m-n)^{-1}\{(m-n)d/(2\pi) + m\epsilon_m - n\epsilon_n\} + m(d/(2\pi)+\epsilon_m)/n \\ &\qquad - \sum_{\nu=n+1}^{m}(v_\nu - v_n)/(n(m-n)) \\ &= (1+m/n)d/(2\pi) + \epsilon_m m^2/(n(m-n)) \\ &\qquad - \epsilon_n n/(m-n) - \sum_{\nu=n+1}^{m}(v_\nu - v_n)/(n(m-n)). \end{aligned}$$

Here we may use inequality (2) of Lemma 1 with

$$p = \epsilon, \quad \mu(n+1) + n \leqq m < (\mu+1)(n+1), \quad \mu = \mu(\epsilon)\to 0 \text{ as } \epsilon \to 0,$$

which gives

$$\limsup_{n\to\infty} v_n/n \leqq (2+\mu)d/(2\pi) + (1+\mu)\epsilon.$$

On allowing here $\epsilon\to 0$ and $\mu\to 0$, we get

$$\limsup_{n\to\infty} v_n/n \leqq d/\pi.$$

In a similar fashion the relation

$$v_m/m = (m-n)^{-1}(V_m/m - V_n/n) + V_n/(nm) + m^{-1}(m-n)^{-1}\sum_{\nu=n+1}^{m}(v_m - v_\nu)$$

yields

$$\liminf_{m\to\infty} v_m/m \geqq d/\pi,$$

whence

$$\lim_{n\to\infty} v_n/n = d/\pi.$$

This proves Lemma 6, and also the first statement of Theorem 4. Assume now

$$\sum_{\nu=1}^{\infty} b_\nu r^\nu \sin \nu\theta_0 \to s(\theta_0) \text{ as } r \to 1-0.$$

Then it is readily seen that

$$(1-r)\sum_{\nu=1}^{\infty}(\nu+1)^{-1}(b_1 \sin\theta_0 + \cdots + \nu b_\nu \sin\nu\theta_0)r^\nu \to 0 \text{ as } r \to 1;$$

hence

$$(1-r)\sum_{\nu=1}^{\infty}(\nu+1)^{-1}(b_1(1-\sin\theta_0) + \cdots + \nu b_\nu(1-\sin\nu\theta_0))r^\nu \to d/\pi \text{ as } r\to 1.$$

Being combined with (9) this yields

$$R_n = \sum_{\nu=1}^{n} S_\nu/(\nu+1) \sim dn/\pi.$$

Taking into account that

$$\sum_1^\infty b_\nu(1-\sin\nu\theta_0) \subset (A)$$

and using the same argument as in the proof of Lemma 6 we conclude

$$S_n \to d/\pi, \text{ or } n^{-1}\sum_{\nu=1}^{n}\nu b_\nu \sin\nu\theta_0 \to 0 \text{ as } n\to\infty.$$

The relation

$$\sum_1^\infty b_\nu \sin \nu\theta_0 = s(\theta_0)$$

now follows by the classical theorem of Tauber.

Conversely, assuming that this relation is satisfied, we have

$$\frac{2}{h}\int_0^h \{\omega(\theta_0+t)+\omega(\theta_0-t)\}dt - s(\theta_0)$$

$$= \left(\sum_{\nu=1}^{n} b_\nu \sin\nu\theta_0 - s(\theta_0)\right) + \sum_{\nu=1}^{n} b_\nu \sin\nu\theta_0\left(\frac{\sin\nu h}{\nu h}-1\right)$$

$$+ \sum_{\nu=n+1}^{\lambda_n} b_\nu \sin\nu\theta_0 \frac{\sin\nu h}{\nu h} + \sum_{\nu=\lambda_n+1}^{\infty} b_\nu \sin\nu\theta_0\frac{\sin\nu h}{\nu h}$$

$$\equiv U_0 + U_1 + U_2 + U_3.$$

Given ϵ, $0<\epsilon<1$, we first choose $N=N(\epsilon)$ so that

$$|U_0| < \epsilon^2, \qquad n > N.$$

We also have

$$|U_1| \leqq \tfrac{1}{6}nh^2\sum_{\nu=1}^{n}\nu|b_\nu| < Cn^2h^2,$$

$$|U_3| \leqq h^{-1}\sum_{\nu=\lambda_n+1}^{\infty}|b_\nu|/\nu < C/(h\lambda_n).$$

Now choose $n=[\epsilon/h]$, $\lambda_n=[\pi/(\epsilon h)]$. An estimate for U_2 and the final result

$$\frac{2}{h}\int_0^h\{\omega(\theta_0+t)+\omega(\theta_0-t)\}dt \to s(\theta_0) \text{ as } h\to 0$$

is obtained by precisely the same argument as in the proof of Theorem 2.

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Massachusetts Institute of Technology,
Cambridge, Mass.